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SOME REMARKS ON THE p-BASIS AND DIFFERENTIAL BASIS

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Abstract. The purpose of this note is to introduce interesting and useful properties differential-basis and p-basis in theorem 3.3. The result gives a existence differential basis of $S^P[\Lambda_1](S)$ which has a p-basis over S^p $(S^p[\Lambda_1])$.

1. Introduction

Let p be always a prime number and all rings are commutative ring with an identity. The concept of *differential-basis* has an important influence on properties of rings, for example the connection with p-basis ([4],[5],[6]). Any differential basis of a regular local ring R of characteristic $p > 0$ over R^p is a p-basis of R over R^p ([4]). Furthermore, we can deduce interesting case as following; Let S be a ring of characteristic p and let Λ be a p-basis of S over S^p and Λ_1 be a subset of Λ, then Λ_1 is a p-basis of $S^p[\Lambda_1]$ over S^p and $\Lambda_2 = \Lambda - \Lambda_1$ is a p-basis of S over $S^p[\Lambda_1]$? The purpose of the present paper is to give an answer to question (Thereom 3.3). For the definition and elementary properties, refer to ([1],[2]).

2. Preliminaries

For any A-module, let $M[x]$ denote the set of all polynomials in x with coefficients in M, that is to say expressions of the form $m_0 + m_1 x + \cdots + m_r x^r (m_i \in M)$.

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PROPOSITION 2.1. *Let the notation be as above ([3]).Defining the product of an element of $A[x]$ and an element of $M[x]$ in the obvious way, we get the following;*

(1) *$M[x]$ is an $A[x]-$module.*
(2) *$M[x] \cong A[x] \otimes_A M$*

Proof. (1) For $a_i \in A, m_j \in M$,

$$(\sum_{i=0}^{n} a_i x^i)(\sum_{j=0}^{m} m_j x^j) = \sum_{k=0}^{m+n} c_k x^k, \quad where \quad c_k = \sum_{i+j=k} a_i m_j$$

since $c_k \in M$, the fact that $M[x]$ is an $A[x]$-module is obvious.

(2) Consider the A-bilinear map $f : A[x] \times M \to M[x]$ defined by

$$f(\sum_{i=0}^{n} a_i x^i, m) = \sum_{i=0}^{n} a_i m x^i.$$

By definition of tensor product, there exists a unique A-linear mapping $f : A[x] \otimes_A M \to M[x]$ such that

$$f'(\sum_{i=0}^{n} a_i x^i \otimes m) = \sum_{i=0}^{n} a_i m x^i$$

A mapping $g' : M[x] \to A[x] \otimes_A M$ defined by

$$g'(\sum_{j=0}^{m} m_j x^j) = \sum_{j=0}^{m} x^j \otimes m_j$$

is a A-linear map.

$$(f' \circ g')(\sum_{j=0}^{m} m_j x^j) = f'(\sum_{j=0}^{m} x^j \otimes m_j) = \sum_{j=0}^{m} m_j x^j$$

$$\begin{aligned}(g' \circ f')(\sum_{i=0}^{n} a_i x^i \otimes m) &= g'(\sum_{i=0}^{n} a_i m x^i) = \sum_{i=0}^{n} x^i \otimes a_i m \\ &= \sum_{i=0}^{n} a_i (x^i \otimes m) \\ &= \sum_{i=0}^{n} (a_i x^i \otimes m).\end{aligned}$$

This implies that $M[x] \cong A[x] \otimes_A M$. □

COROLLARY 2.2. *For A-module M,*

(1) $M[x_1, \cdots, x_n] : A[x_1, \cdots, x_n] - module.$
(2) $M[x_1, \cdots, x_n] \cong A[x_1, \cdots, x_n] \otimes_A M.$

Proof. (1) It is obvious.

(2) Induction on n. For $n = 1$, it was proved by proposition 2.2. By inductive hypothesis,

$$
\begin{aligned}
&M[x_1, \cdots, x_{n-1}] \cong A[x_1, \cdots, x_{n-1}] \otimes_A M. \\
&A[x_1, \cdots, x_n] \otimes_A M \\
&= (A[x_1, \cdots, x_{n-1}][x_n] \otimes_{A[x_1, \cdots, x_{n-1}]} A[x_1, \cdots, x_{n-1}]) \otimes_A M \\
&\cong A[x_1, \cdots, x_{n-1}][x_n] \otimes_{A[x_1, \cdots, x_{n-1}]} (A[x_1, \cdots, x_{n-1}] \otimes_A M) \\
&\cong A[x_1, \cdots, x_{n-1}][x_n] \otimes_{A[x_1, \cdots, x_{n-1}]} M[x_1, \cdots, x_{n-1}] \\
&\cong M[x_1, \cdots, x_{n-1}][x_n].
\end{aligned}
$$

□

3. Main theorem

Let k be a ring, A a k-algebra and $B = A \otimes_k K$. Consider the homomorphism of k-algebras $\epsilon : B \to A$ and $\lambda_1, \lambda_2 : A \to B$ defined by $\epsilon(a \otimes a') = aa', \lambda_1(a) = a \otimes 1, \lambda_2(a) = 1 \otimes a$. Once and for all, we make $B = A \otimes A$ an A-algebra via λ_1. We denote the kernel of ϵ by $I_{A/k}$ or simple by I, and we put $I/I^2 = \Omega_{A/k}$. The B-module I, I^2 and $\Omega_{A/k}$is called the *module of differentials (or of Kahler differential)* of A over k.We have $\epsilon\lambda_1 = \epsilon\lambda_2 = id_A$. Therefore, if we denote the natural homomorphism $B \to B/I^2$ by ν and if we put $d^* = \lambda_2 - \lambda_1$ and $d = \nu d^*$, then we get a derivation $d : A \to \Omega_{A/k}$.

From above definition, we have following Lemma. We introduce some properties without proofs.

LEMMA 3.1. *(1) If*

$$
\begin{tikzcd}
k \arrow[r, "f"] \arrow[d, "g"'] & k' \arrow[d, "g'"] \\
A \arrow[r, "f'"'] & A'
\end{tikzcd}
$$

is commutative diagram of rings and homomorphisms, then there is a natural homomorphisms of A-modules $\Omega_{A/k} \to \Omega_{A'/k'}$, *hence also a natural homomorphisms of A'-modules* $\Omega_{A/k} \otimes_A A' \to \Omega_{A'/k'}$.

(2) If $A' = A \otimes_k k'$ in (1), then the last homomorphism is an isomorphism :

$$\Omega_{A'/k'} = \Omega_{A/k} \otimes_k k' \to \Omega_{A/k} \otimes_A A'.$$

DEFINITION 3.2. Let S be a ring of characteristic p and S^p denote the subring $\{x^p | x \in S\}$ and let S' be a subring of S. A subset Γ of S is said to be $p-$ *independent* over S', if the monomials $b_1^{e_1} \cdots b_n^{e_n}$ where $b_1, \cdots, b_n$ are distinct element of Γ and $0 \leq e \leq p-1$, are linearly independent over $S^p[S']$.

If $B \subset K$ is p-independent over k and $K = K^p(k, B)$, we say that B is a *p-basis* of K/k. If $C \subset K$ is p-independent over k, then there exists a p-basis of K/k containing C.

LEMMA 3.3. *Let S be a ring of characteristic p and S' a subring of S containing S^p and let $\{x_1, \cdots, x_n\}$ be a subset of S. If $\{dx_1, \cdots, dx_n\}$ is S-free in $\Omega_{s/s'}$, then $\{x_1, \cdots, x_n\}$ is p-independent over S'.*

Proof. If $x_1, \cdots, x_n$ are not p-independent over S', we can take a reduced polynomial $f(X_1, \cdots, X_n) \in S'[X]$ of lowest degree such that $f(x_1, \cdots, x_n) = 0$.Then

$$df(x_1, \cdots, x_n) = \sum_{i=1}^{n} (\frac{\partial f}{\partial x_n}) dx_n = 0 \text{ in } \Omega_{s/s'} \text{ and for some i, } \frac{\partial f}{\partial x_i} \neq 0.$$

If for all i, $\frac{\partial f}{\partial x_i} = 0$,

$$0 = \sum_{i=0}^{n} (\frac{\partial f}{\partial x_i}) \overset{put}{=} g(x_1, \cdots, x_n) = 0.$$

Then $deg(g) < deg(f)$, it is contradiction. Thus, $x_1, \cdots, x_n$ is a $p-$ indenpendent over S'. $\square$

THEOREM 3.4. *Let S be a ring of characteristic p and let Λ be a differential basis of S over S^p and Λ_1 be a subset of Λ, then Λ_1 is a differential basis of $S^p[\Lambda_1]$ over S^p and $\Lambda_2 = \Lambda - \Lambda_1$ is a differential basis of S over $S^p[\Lambda_1]$*

Proof. **Step (1)**

$$\begin{tikzcd} S^p[\Lambda_1] \arrow[r, "f"] \arrow[d, "g"'] & S \arrow[d, "g'"] \\ S^p \arrow[r, "f'"'] & S^p \end{tikzcd}$$

is commutative diagram of rings and homomorphism, by (1) of Lemma 3.1, we have natural homomorphism

$$\Omega_{S^p[\Lambda_1]/S^p} \to \Omega_{S/S^p}.$$

$\Omega_{S^p[\Lambda_1]/S^p}$ is generated by $\{d(x_\lambda) \mid x_\lambda \in \Lambda_1\}$ as $S^p[\Lambda_1]-$ module. This is clear since d is a derivation. Also $\Omega_{S^p[\Lambda_1]/S^p}$ is a free $S^p[\Lambda_1]-$ module with $\{d(x_\lambda) \mid x_\lambda \in \Lambda_1\}$ as a basis. In fact, suppose $\sum a_\lambda d(x_\lambda) = 0 \quad (a_\lambda \in S^p[\Lambda_1])$. Applying h to $\sum a_\lambda d(x_\lambda) = 0$ we see $\sum a_\lambda d(x_\lambda) = 0$, in Ω_{S/S^p}. Since $\Lambda_1 \subset \Lambda$, $\{d(x_\lambda) \mid x_\lambda \in \Lambda_1\}$:linear independent over S. Thus, $a_\lambda = 0$. Therefore Λ_1 is differential basis of $\Omega_{S^p[\Lambda_1]/S^p}$.

Step (2) Since $\{\Lambda\}$ is differential basis of Ω_{S/S^p}, by Lemma 3.2,$\{\Lambda\}$ is $p-$indepent over S^p. This implies that $\Lambda_1(\subset \Lambda)$ is $p-$ independent over S^p. i.e, $\{x_1^{e_1} \cdots x_n^{e_n} \mid 0 \le e_i < p\}$ are linearly independent over S^p (where $x_1, \cdots, x_n$ are distinct elements of Λ_1). Then, by (2) of corollary 2.2, $S = S^p[\Lambda_1] \cong S^p[\Lambda_1] \otimes_{S^p} S$. By (2) of Lemma 3.1,

$$\Omega_{S/S^p} \otimes_S S \cong \Omega_{S/S^p} \xrightarrow{h'} \Omega_{S/S^p[\Lambda_1]}$$

is an isomorphism as $S-$module. The homomorphism h' is defined by

$$\begin{aligned} & h'(\sum a_\lambda d_{S/S^p}(x_\lambda)) \\ & = \sum a_\lambda d_{S/S^p[\Lambda_1]}(x_\lambda) \quad (a_\lambda \in S, x_\lambda \in \Lambda, d_{S/S^p[\Lambda_1]}(x_\lambda) = 0 \ \text{ for } \ x_\lambda \in \Lambda) \\ & = \sum a_\mu d_{S/S^p[\Lambda_1]}(x_\mu)(a_\mu \in S, x_\mu \in \Lambda_2). \end{aligned}$$

$\Omega_{S/S^p[\Lambda_1]}$ is generated by $\{d(x_\mu) \mid x_\mu \in \Lambda_2\}$ as $S-$module. In fact, $\{d_{S/S^P[\Lambda_1]}(x_\mu) \mid x_\mu \in \Lambda_2\}$ is linearly independent over S.

For

$$\sum a_\mu d_{S/S^p[\Lambda_1]}(x_\mu) \in \Omega_{S/S^p[\Lambda_1]},$$

there exist a $\sum a_\mu d_{S/S^p}(x_\mu)$ such that

$$h'(\sum a_\mu d_{S/S^p}(x_\mu)) = \sum a_\mu d_{S/S^p[\Lambda_1]}(x_\mu).$$

if $\sum a_\lambda d_{S/S^p[\mu]}(x_\mu) = 0$ in $\Omega_{S/S^p[\Lambda_1]}, \sum a_\mu d_{S/S^p}(x_\mu) = 0$ in Ω_{S/S^p}. It implis that $a_\mu = 0$. Thus,

$$\{d_{S/S^P[\Lambda_1]}(x_\mu) \mid x_\mu \in \Lambda_2\}$$

is linearly independent over S. Therefor the assertion ends.

□

References

[1] H. Matsumura, *Commutative algebra*, Benjamin, 1970 (second edn. 1980).
[2] H. Matsumura, *Commutative Ring Theory*, Cambridge Univ., 1989.
[3] M. F. Atiyah and I. G. Macdonal, *Introduction to commutative algebra.*
[4] T. Kimura and H. Niitsma, *Differential basis and p-basis of a regular local ring*, Proc. Amer. Math. Soc. **92** (1984), no. 3, 335-338.
[5] Crupi and Marilena, *Free modules of differentials and p-basis*, Atti Sem. Mat. Fis. Univ. Modena **42** (1984), no. 1, 141-157.
[6] L. CHiantini, *p-basi e basi differenziali di un anello*, Rend. Univ. Pol. **37** (1979), no. 2, 103-121.

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